The Old Liar

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1 The Problem

In this paper I am going to discuss and solve the following problem:

> Three friends, Adam, Beth, and Cleo, are sitting together and talking. Together they are 150 years old.
>
> Adam: In 9 years, you two together will be as old as we three together are today.
>
> Beth: Ten years ago, the sum of your both ages was my age right now.
>
> Cleo: Beth, 10 years ago I was twice your age.
>
> If we know that the oldest, and only the oldest of the three is a liar, can you tell the ages of the three persons?

2 The Solution

Let a denote the age of Adam, b the age of Beth, and c the age of Cleo at the day they meet. Two interpretations are possible: The first one is that the numbers are the rounded down natural numbers one usually gives when asked for the age. The second interpretation is that they give the age exactly, as real numbers. Mathematically there is no difference, except that in the first version some possible solutions would be rejected for being not natural.

The given information to be together 150 years old translates into the equation

$$a + b + c = 150. \tag{1}$$

Adam's statement translates into

$$(b + 9) + (c + 9) = 150. \tag{2}$$

Keep in mind that in 9 years, Beth and Cleo are $b+9$ respectively $c+9$ years old. We certainly have to assume that they talk about in *precisely* 9 years. Beth's statement reads as

$$(a - 10) + (c - 10) = b, \tag{3}$$

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$$(c-10) = 2(b-10), \tag{4}$$

if we assume that "I was twice your age" means "twice your age then".

2.1 The tedious approach

I would like to demonstrate an approach by which problems like this can be treated. In the next Section I will, however, use a slightly different, more elegant, approach, which, in combination with Case 1 below, solves our problem completely.

By the information given we know that equation (1) is true, but only exactly two of the equations (2), (3), and (4) are true. One obvious approach now would be to consider the three possible cases separately, solve the resulting three equations with three variables, and check in each case about consistency like whether the excluded equation is false and really corresponds to the largest variable in the solution.

Case 1: Assume Beth is the oldest. We solve the system consisting of equations (1), (2), and (4). We use equation (1) and then equation (2) to get

$$150 - a = b + c = 150 - 9 - 9 = 132, \tag{5}$$

and therefore $a = 18$. Substituting this into equation (1) yields

$$b + c = 132, \tag{6}$$

and equation (4) can be reformulated as

$$2b - c = 10. \tag{7}$$

By adding equations (6) and (7) we obtain $b = \frac{142}{3}$ and $c = \frac{254}{3}$, a contradiction to the leading assumption of Beth being the oldest in this Case 1 (even when one allows fractions as ages).

Case 2: In the same way one can treat the case where one assumes that Beth is the oldest. Then we solve the equation system consisting of equations (1), (3) and (4). We substitute equation (3) into equations (1) and (4) to get

$$a + ((a-10) + (c-10)) + c = 150 \tag{8}$$
$$(c-10) = 2((a-10) + (c-10) - 10). \tag{9}$$

We simplify both equations to get

$$2a + 2c = 170 \tag{10}$$
$$2a + c = 50. \tag{11}$$

By subtraction equation (11) from equation (10) we get $c = 120$. Substituting this into equation (11) yields $a = -35$, at which point we can stop to treat this case—negative age is impossible.

Case 3: Cleo is the oldest. The resulting solution is $a = 18$, $b = 65$, $c = 67$. So the solution is consistent with the case assumption, and all solution values are natural numbers as desired, therefore this is the solution to our problem.

2.2 A more elegant approach

It is interesting to see that, if we believe Beth, we already know her age. Since if she is not the liar, we can combine equations (1) and (3) to get

$$150 - b = a + c = b + 20. \tag{12}$$

Therefore $b = 65$ under this assumption of Beth being not the oldest. But then Cleo's equation (4) cannot be true, since an age of 120 years would follow for Cleo, and Adam would be -35 years by equation (1). Therefore believing Beth implies believing Adam too, and by (2) there follows $c = 67$. Using (1) we get $a = 18$.

All what needs to be checked is the case where we don't believe Beth. It is just Case 1 in Subsection 2.1.

3 The Solution

The only solution to the problem is that Adam is 18 years old, Beth is 65, and Cleo is 67 years old, and therefore lying.

4 Generalizations

It is easy to construct similar problems, just by letting Adam, Beth and Cleo formulate sentences about the relation of their ages. The approach described in Subsection 2.1 can be used in any case to find the solutions, provided there are such. Also the extension to more than three persons is as straightforward as is the extension of the solution procedure described.