Dividing Six Pralines

March 2010

Abstract: The procedure to divide a cake such that everybody gets what he or she considers the better half is rather well-known. But what if we have a given number, say six, of pralines to divide among two persons, and these pralines cannot be cut? We will consider a few very simple methods to divide these pralines and compare their fairness, equity, and efficiency using simulations.

You have two daughters, Ann and Beth. A friend gave you a box with six pralines for them. They are all different, and include some strange flavors. You don't know which one of your daughters prefers which praline, but it is your task to divide them among your daughters. You want to be fair, and you want them to be as happy as possible. What would you do?

We operate with two important assumptions. The first one is that the two daughters know which one of the six pralines they like, and which one not, and also how much. We also assume that the two daughters are so close to each other that they also know the preference of their sister. This second assumption will be dropped later, in the last section.

Since you don't have a clue about what Ann and Beth prefer, but Ann and Beth do, the simplest way would be to ask them which pralines they prefer and give them the pralines based on this information. Of course, they might prefer the same pralines and, therefore, exaggerate. So how sure could you be then that they were telling the truth?

For this reason it might be better to let them decide about the distributions themselves. You could set up a distribution procedure, and ask them to act according to the rules described in the procedure and distribute the pralines in this way. For instance, we could let them alternately select one praline each, until all are chosen. This procedure will later be investigated in detail, under the name "'ABABAB"'. Such procedures are called 'games' in the literature and throughout the paper. Then your task is not to play the game successfully—this is the task of the 'players' Ann and Beth. Rather your task is to design a 'good' game, a procedure that will very likely produce a fair distribution that makes Ann and Beth as happy as possible.

In this paper we will present some very simple possible games, and will discuss some of their features, like simplicity, equity, and efficiency. In Section 1 we discuss these features. In Section 2 we introduce and analyze these games. In Section 3 we compare them, using simulations for different assumptions on the distribution of the values. In Section 4 we have a brief look into what would happen if the two players would **not** know each other well.

Looking at our concrete example of six items and two players, the purpose of this paper is to give a concrete introduction into sequential game theory with perfect information, its main tool of backward induction. I have used simpler versions of this topic in introductory Game Theory classes as group projects, but think the topic also fits very well in other General Education Mathematics classes.

1 Fairness, Equity, Efficiency, and other Features

Before we look at the different games, we need to discuss the assumptions on the payoffs in more detail, and we need to discuss fairness, equity, and efficiency.

1.1 Assumptions

Throughout this paper, let $A(1), A(2), \ldots A(6)$ denote the values Ann assigns to the six pralines, and let $B(1), B(2), \ldots B(6)$ be the values they have for Beth.

The first assumption is that the payoff for the player is the sum of the values she attaches to the pralines she gets. Different to other goods, there are no pairs of pralines that have only worth in combination with each other, or pralines whose possession would diminish the value of possessing others.

The next question is whether values of different persons can be related. Assume Ann assigns the value of 1 to all six pralines, and Beth assigns the value of 10 to all of them. It seems like Ann values them little, and Beth much. Wouldn't it be better for the society as a whole if Beth gets all six pralines? But maybe standards are different and not comparable. Maybe somebody giving a value of 10 to a praline just doesn't know what is possible in praline quality.

In this paper we will assume that we can not compare the values of Ann and Beth. What we do instead is defining a level of 'satisfaction' between 0% and 100% for each player, 100% being the sum of the values of all six pralines for that player. Therefore the **satisfaction** for a player would be the ratio of total value received and total possible value for that player.

For a different formulation, we look at the relative values defined by $a(i) = A(i)/(A(1) + \cdots A(6))$ and $b(i) = B(i)/(B(1) + \cdots B(6))$. Note that the sum of all relative values of each player equals 1. Then the satisfaction of a player is the sum of the player's relative values of all received pralines.

We also assume that the players play rational. That means that they try to maximize their payoff, and also anticipate the others moves based on this assumption. Rational play also implies that the players are capable and willing to do the sometimes complicated analysis.

1.2 What outcomes does society prefer

The players want to maximize their satisfaction. But we, as parent, have two or three different objectives. First, we would like to see the satisfaction of both our daughters as high as possible. Again, for simplicity we concentrate on the sum of both satisfactions. But to evaluate the quality of the outcome, more important than the value of this sum of satisfactions would be how much more could be achieved. Let us call the difference between possible total satisfaction that can be achieved with the given preferences and the actual total satisfaction the **inefficiency** (not be confused with the notion of Pareto-inefficiency used in the literature) of the outcome. The lower it is, the better.

A second objective might be fairness and equity. We want both daughters to get about the same satisfaction. We call the absolute value of the difference of both satisfactions the **inequity** of the outcome. The lower it is, the better. Again what can be achieved depends on the preferences, but different to the concept of inefficiency, here we don't compare with what could be achieved.

A third interesting measure is the **minimum satisfaction** of the two players. Having a high such minimum would imply that both are to some extend satisfied. This measure, which is also discussed in [BS 2006] and [BD 2005], could be seen as a blend of efficiency and equity, since among different outcomes with the same efficiency, higher minimum satisfaction means higher equity. And among different outcomes with the same equity higher minimum satisfaction means higher efficiency.

Note that after you have distributed the six pralines, you will not be able to compute satisfaction, inefficiency, and equity, since you still don't know the values Ann and Beth assign to the pralines, unless they are willing to reveal them truthfully afterwards. Therefore, usually you will not be able to determine whether the result achieved is 'good' or not. Therefore the question which distribution game is best can not be resolved by running experiments with real people.

1.3 Santa Claus Procedures

Let us discuss briefly how someone, who, like Santa Claus, knows the values $A(1), \ldots A(6)$, $B(1), \ldots B(6)$ of the pralines for Ann and Beth, could compute the optimal values discussed above

We achieve the maximum sum of satisfactions by giving each praline to the player who values it relatively more. Since the sum of all 12 relative values equals 200%, and each one of the six relative values of the assigned pralines is larger than the corresponding relative value of that praline for the other player, the sum of satisfactions achieved in this way lies between 100% and 200%.

Let me give an example. Assume $A(1) = 1, A(2) = 2, A(3) = 3, A(4) = 4, A(5) = 5, A(6) = 5$, and $B(1) = 11, B(2) = 4, B(3) = 6, B(4) = 9, B(5) = 12, B(6) = 8$. Then the relative values are $a(1) = 5\%, a(2) = 10\%, a(3) = 15\%, a(4) = 20\%, a(5) = 25\%, a(6) = 25\%$, and $b(1) = 22\%, b(2) = 8\%, b(3) = 12\%, b(4) = 18\%, b(5) = 24\%, b(6) = 16\%$. Therefore Ann gets pralines 2, 3, 4, 5, and 6, and Beth gets only praline 1. Ann's satisfaction is 95%, and Beth's 22%, therefore the total satisfaction for this distribution is 117%.

Comparing all $2^6 = 64$ possible distributions of the six pralines, we can compute a distribution with minimum inequity, or one with maximum minimum satisfaction. In the example above, the distribution where Ann gets pralines 2, 4, and 6 and Beth the remaining ones, gives a satisfaction of 55% to Ann and of 58% to beth and minimizes inequity (3%) and maximizes minimum satisfaction (55%). Note that these distributions can be different, and also different to that maximizing sum of satisfactions.

There are examples of preferences where inequity is always close to 100% and minimum satisfaction is always close to 0%, namely cases where there is one (caviar?) praline that both players value much higher than the other five.

2 The Games

In this section we describe a handful of distribution games as well as the technique of 'backward induction' used to analyze sequential games.

2.1 Choosing One-By-One Games

A very simple idea is to let the two players choose items one-by-one. The players may alternate, but by using other choice sequences, we arrive at different games.

- **ABABAB:** Starting with Ann, both alternate of selecting one praline until all are chosen.
- **ABBABA:** First Ann chooses one praline, then Beth chooses two, after that Ann and Beth alternate selecting one.
- **ABBAAB:** First Ann chooses one praline, then Beth chooses two, then Ann chooses two, and Beth gets the remaining one.

In each of these games, Ann and Beth move alternately, but a move may consist in choosing more than one praline. Since taking the remaining praline at the end is not really a move, since there is no choice, the game 'ABABAB' has five moves, 'ABBABA' has four moves, and 'ABBAAB' has 3 moves.

Wouldn't it be rather obvious how Ann and Beth would play? In game 'ABABAB', for instance, wouldn't Ann just choose the praline that is most valuable for her, and then Beth choose the remaining praline with highest value for her, and so on? Deciding in this manner is called 'greedy strategy', for obvious reasons, and indeed that may happen if both players don't think too seriously about what to do, if they are small children, or if they don't care too much about the outcome. But if the stakes are high and the players sophisticated, they may think more deeply about their decisions, and may indeed deviate from the greedy strategy.

Let's give an example: We assume that game 'ABABAB' is played, that three pralines have already been chosen, let's say pralines 4, 5, and 6, and that Ann and Beth attach the values $A(1) = 2, A(2) = 3, A(3) = 1$, and $B(1) = 1, B(2) = 2, B(3) = 3$ to the remaining three pralines. What would Beth choose in step 4 of the game? If Beth applies the greedy strategy and chooses praline number 3 of the highest value of 3 to her, then Ann would chooses praline 2 after, and Beth get the remaining praline number 1 of value 1 to her. In this way Beth would add the value of 4 to her payoff. But Beth could also make a tactical decision and choose praline number 2 in step 4. Then Ann would have to take praline number 1 and Beth would eventually also get praline number 3, improving her payoff by the value of 5. Although Ann would be furious over Beth's previous move, playing rationally she still has no reason not to chose her best choice in her last move. Rational play includes that players only hurt other players if they get an own advantage.

2.2 Analyzing sequential games using backward induction

Given the values Ann and Beth assign to the six pralines, we want to know how Ann and Beth would move in the first, second, third move, and so on. We choose this concrete game 'ABBAAB' to explain a general procedure, called 'backward induction' which is used to analyze any sequential game. Take the example $A(1) = 1, A(2) = 2, A(3) = 3, A(4) = 4, A(5) = 5, A(6) = 6$, and $B(1) = 6, B(2) = 2, B(3) = 1, B(4) = 4, B(5) = 3, B(6) = 5$. In the game 'ABBAAB', how would Ann decide in the first move? How would Beth decide in the second move? And how would Ann decide in the third move? We may not know the answer to the first question, but for the second and third question it is rather obvious: It depends. What Beth will do in the second move depends on what Ann did in the first move, and how Ann will decide in the last move depends on what Ann and Beth have chosen in the first two moves.

For this reason we need the definition of a **position**. Whenever you decide, you look at your options, but also take all other information available into account. In the game 'ABBAAB', when Ann makes the first move, nothing has happened yet, so there is just one position. This single position at the very beginning of a (sequential) game is called the **start position**. When Beth is about to make her move, the second move of the game, Ann could have chosen any of the six pralines in her first move. This choice of Ann determines the position of Beth at move 2, therefore she has six of them. Then Beth chooses 2 pralines out of the remaining five. At the third move of the game, when Ann has to choose again, Ann has chosen any of six possibilities, and Beth has chosen two out of the remaining 5 pralines. Thus Ann can face any of $6 \cdot 10 = 60$ positions at that stage. Finally, after Ann has moved again and only one praline is left, there are $60 \cdot 3 = 180$ different positions, where Beth takes the remaining one and the game is essentially over We call these the **end positions**.

In every non-end position, one player has to move. All positions that can result from this position by the player making a choice are called *successors* of that position.

We want to analyze each one of these 1 + 6 + 120 + 180 positions. By 'analyzing' we mean that for each not-end position, the player about to move has to be told how to move. Even more important, we will see that for each one of these positions we can tell how much Ann and Beth will eventually receive when facing this position, provided both play rationally.

The main idea is rather simple. If we consider any position, let's name it position P, and all its successors are analyzed, then our position P can be analyzed as well. Remember that each of these resulting positions has values for Ann and Beth attached which are supposed to be the resulting payoffs when facing this position. The player who has to move in position P will select that position, let's name it position Q, where the attached value for that player is highest, and position P will inherit both these values from position Q, since whenever position P occurs, a rational player will move to position Q.

In all games considered in this paper, positions can be grouped into rounds, and each position leads to positions in the next higher round. In that case we start in the highest round, where all positions are end positions and therefore have already values for Ann and Beth attached. Then we analyze all positions in the next lower round, and so on, until we face the start position in the first round.

In the example given above, let '6,45' denote the position where Ann has chosen praline 6 in the first move and Beth pralines 4 and 5 in the second move. In this position, Ann would chose pralines 2 and 3, since the resulting end position '6,45,23' has highest payoff of 11 for Ann (and a payoff of 13 for Beth). On the other hand, position '6,14' has a value of 14 for Ann and only 12 for Beth. Thus, if Ann starts by selecting praline 6, the greedy decision to chose pralines 1 and 4 would be worse than selecting pralines 4 and 5. By considering all positions, one can see that that's exactly what would happen: Ann would select praline 6, then Beth would select pralines 4 and 5, then Ann would select pralines 2 and 3, and Beth would take the remaining praline 1 (with the highest value to her) at the very end.

This procedure discussed is known in the literature under the name **backward induction**.

Let's now briefly discuss how good these three games considered are in the worst case. If Ann plays greedy in 'ABABAB', she will always be able to get her first, third, and fifth choice. This way, she can achieve 50% satisfaction, since the values of these choices are greater or equal to her values of the second, fourth, and sixth choice, respectively. Note that Ann may not play the greedy strategy, but it is an option, and Ann will only deviate if she gets more. Thus Ann will always be able to get at least 50% satisfaction in the game 'ABABAB'. In the games 'ABBABA' or 'ABBAAB', Ann can get her first and fourth choice (and one more, either the sixth or the fifth). Since her first choice is larger or equal to her second and third, and her fourth larger or equal to her fifth and sixth, she can always get at least a satisfaction of 100/3 %.

Beth, on the other hand, has no satisfaction guarantee in any of these three games, since it could be that she essentially only values one item, which may be taken by Ann in her first move.

The following two examples show that the bounds are tight. Assume $a(1) = 17\%, a(2) = a(3) = a(4) = a(5) = a(6) = 16.6\%$, and $b(1) = 100 - r\%, b(2) = b(3) = b(4) = b(5) = b(6) = s\%$, with $r = 5s$ very small values. Assigning praline 1 to Beth and all others to Ann, Ann would get a satisfaction of 83%, and Beth even of $100 - r\%$, but playing 'ABABAB', Ann will get 50.2 % and Beth almost nothing, just $3s\%$.

For the other example, take $a(1) = 34\%, a(2) = a(3) = 33\%, a(4) = a(5) = a(6) = 0\%$, and $b(1) = 100 - r\%, b(2) = b(3) = s\%, b(4) = b(5) = b(6) = 0\%$, with $r = 2s$ very small values. Then 166% total satisfaction is possible, but playing 'ABBABA' or 'ABBAAB', Ann will only get 34%, and Beth will only gain as little as $2s\%$.

Thus, looking only on worst cases, game 'ABABAB' may be considered better than the games 'ABBABA' and 'ABBAAB', as far as total satisfaction is concerned.

2.3 Cut & Choose

The next game is the discrete version of the famous so-called 'cake cutting' procedure used in the continuous version in [S 1949]:

- **Cut & Choose (C&C):** First Ann divides the six pralines into two heaps. Then Beth decides which heap she wants, and Ann gets the remaining heap.

Again it is a sequential game, with Ann performing the first move and Beth the second. The reasoning in this game is very common sense. Beth in her second move will always choose the heap with the greatest value to her. This, of course, Ann is anticipating, therefore out of the 32 possible divisions into two heaps, Ann will choose that where the remaining heap that Beth wouldn't choose has greatest value to her.

An obvious advantage of this game over the three games discussed in Subsection 2.1 is its simplicity. It is sequential, but has only two steps. Therefore tactical play is obvious, and the analysis of the game is not difficult.

In the 'cake cutting' version with divisible goods, Ann can always select a partition where both parts have equal weight for Ann. Thus Ann can expect at least 50% satisfaction. If she knows Beth's preferences, Ann may be able to get more, by offering a partition where the two parts are worth 51% and 49% for Beth, and where the part of lower value to Beth has a high value to Ann. Therefore, in the continuous version, both Ann and Beth will always achieve a relative satisfaction of at least 50% (with Beth's satisfaction not exceeding 50% much). Although this is impossible to achieve for indivisible goods, Beth will get a satisfaction of at least 50%.

The following example shows that in the worst case, the total satisfaction will be close to 50%, even in cases where a total satisfaction of about 150% is possible. Let $a(1) = 100 - t\%, a(2) = a(3) = a(4) = a(5) = a(6) = u\%$, and $b(1) = 50 + r\%, b(2) = b(3) = b(4) = b(5) = b(6) = 10 - s\%$, with r, s, t, u very small values obeying $t = 5u$ and $r = 5s$. Since Ann knows that Beth will take the heap containing praline 1, she will put praline 1 in one heap and pralines 2 to 6 in the other. The satisfaction for Ann is 50+r %, and that for Beth 5u %. On the other hand, by giving praline 1 to Ann and the five others to Beth, a satisfaction of 100-t % could be achieved for Ann, and a decent satisfaction of 50-5s % for Beth.

2.4 Random & Exchange

Maybe the most obvious procedure is to give each daughter three pralines, and let them handle the situation themselves by trading. Of course, the rules have to be made more precise. One of the many ways to formalize this is as follows:

- **Random & Exchange (R&E(n)):** Each player gets three pralines randomly. Then n rounds are played, where one player tells the other which two pralines she would like to exchange. The other player can agree to the exchange, but can also disagree. Naturally, the exchange is only performed if that player agrees. In the first round, Ann makes the exchange proposal, and then the proposer role alternates.

At first sight, this looks like a very good procedure, at least if we allow many rounds. There is ample opportunity to fix any inefficiencies the random assignment has made. However, as we will see later when we look at data, allowing more rounds doesn't really make much of a difference. With more rounds, the players are just more hesitant to accept an exchange in an earlier round, and we see more tactical play.

The next procedure is similar, but allows a little more flexibility by breaking the exchange proposals into two subproposals made by both players:

- **Random & Double Exchange (R&DE2):** Each player gets three pralines randomly. Then Ann tells Beth which one of Beth's pralines she would like to have. Beth tells Ann which one of Ann's pralines she would like to have in exchange for that. If both agree with the deal, the two pieces are exchanged. Then a second round of possible exchanges follows, this time Beth proposing one of Ann's pralines first, followed by Ann proposing one of Beth's pralines, again followed by an exchange of these items if both agree.

These games are again sequential and can be analyzed using backward induction, but both contain randomness at the very beginning.

3 Simulations

To see how good these games are, results that give guarantees for the games in the worst case are certainly interesting. Above, such guarantees have been given for the three 'selecting one-by one' games and for 'Cut and Choose'. But we are more interested in finding out how these games perform on average.

To test the performance of the games considered, we simulated 2000 different preferences that were created randomly. To be more precise, the values for the six pralines for Ann and Beth were created as 12 independent random variables with uniform distribution in the interval $[0, 1]$. Roughly speaking that means that each number between 0 and 1 is equally likely to occur as value of each praline. A little more precisely, for every integer t, each decimal number between 0 and 1 with t decimals is equally likely to occur as a value for each praline' if we round to t decimals. We also used a distribution method—not really a game since nobody decides anything— called 'Random', consisting in giving Ann three pralines randomly, and the three remaining ones to Beth. The results were as shown in Table 1:

Method	average Ann's sat	average Beth's sat	average sum of sat	average in-efficiency	average inequity	average min sat
Random	49.5%	50.1%	99.7%	32.6%	14.4%	42.6%
ABABAB	67.5%	59.4%	126.9%	5.4%	12.1%	57.4%
ABBABA	63.4%	63.1%	126.5%	5.7%	10.0%	58.3%
ABBAAB	65.2%	61.6%	126.7%	5.5%	10.6%	58.1%
C&C	71.9%	56.0%	127.8%	4.4%	16.3%	55.8%
R&E2	61.1%	64.5%	125.6%	6.6%	14.0%	55.8%
R&E12	61.2%	64.5%	125.7%	6.5%	14.0%	55.9%
R&DE2	62.6%	62.9%	125.5%	6.7%	14.1%	55.7%

Table 1: Average values for 2000 random simulations

On average the possible sum of Ann's and Beth's satisfaction was 132.2%, the average minimum inequity was 1.7%, and the possible minimum of both satisfactions was 61.5%. All these values were computed using the three Santa Claus procedures discussed above. Recall that they are theoretical, and since there may not be a Santa Claus, they don't help us parents to distribute the pralines.

All but the random method went relatively close to these theoretical optima for sum of satisfaction and minimum satisfaction, but not for inequity.

In Figure 1 the efficiencies versus equities are displayed for the seven methods. Obviously ‘C&C’ is more efficient than ‘ABABAB’, which itself is more efficient than ‘ABBABA’ or ‘ABBAAB’. However, the equity increases when moving from ‘C&C’ over ‘ABABAB’ to ‘ABBABA’ or ‘ABBAAB’. For these methods there seems to be a tradeoff between efficiency and equity.

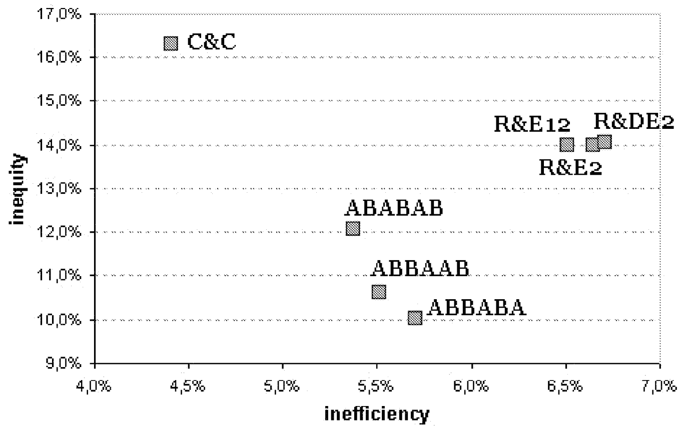

Figure 1: Efficiency versus Equity

Since the numbers depend on random input data for the games, you may ask how random the result is. This question can be answered using some statistics. In particular one needs the standard deviations for the different values. This standard deviation expresses how much spread one can expect around the average.

For instance, the average inefficiency for the 2000 cases of C&C is $\mu(\text{C\&C}) = 4.4\% = 0.044$ with standard deviation $\sigma(\text{C\&C}) = 0.051$. Then we know that the ‘true’ average inefficiency of C&C lies between $\mu(\text{C\&C}) - \frac{2\sigma(\text{C\&C})}{\sqrt{2000}}$ and $\mu(\text{C\&C}) + \frac{2\sigma(\text{C\&C})}{\sqrt{2000}}$, i.e. between 4.18% and 4.63%. with 95% probability.

Doing the same for ‘ABABAB’ with $\mu(\text{ABABAB}) = 5.4\% = 0.054$ and $\sigma(\text{ABABAB}) = 0.065$, with 95% probability the true average inefficiency of ‘ABABAB’ lies between 5.08% and 5.67%.

Therefore ‘C&C’ is more efficient than ‘ABABAB’ with significance 95%. In the same way we can conclude that ‘ABABAB’ is almost surely more efficient than ‘ABBABA’ and ‘ABBABA’ is almost surely more efficient than the ‘Random & Exchange’ games. But between ‘ABBAAB’ and ‘ABABAB’ or ‘ABBABA’ one cannot draw a significant conclusion, the same as within the ‘Random & Exchange’ group.

As for the equity, method ‘ABBABA’ achieves better equity than all others except maybe ‘ABBAAB’, again with 95% significance.

One could also graph our third parameter—maximum minimum satisfaction, or rather its difference to the optimal value, as shown in Table 2— versus efficiency and equity. If we graph these values versus efficiencies, the graph looks almost like Figure 1. The values versus inequity can be seen in Figure 2.

From the graph it is rather obvious that both concepts seem to be heavily related, since the points lie almost on a straight line. We need another statistical concept: The **correlation coefficient** expresses how close a relation between a sequence of values and another sequence of values is to a linear one. The correlation coefficient lies always between -1 and 1. The closer this correlation coefficient is to 1, the closer the data points lie on a straight line with positive slope, and the closer it is to -1 the closer the data points lie on a straight line with negative slope. Here we obviously expect a correlation coefficient close to 1. If one does the calculations, one gets a value of 0.93 (with the 'Random' method excluded from consideration. If this method is included, the correlation coefficient drops to 0.42).

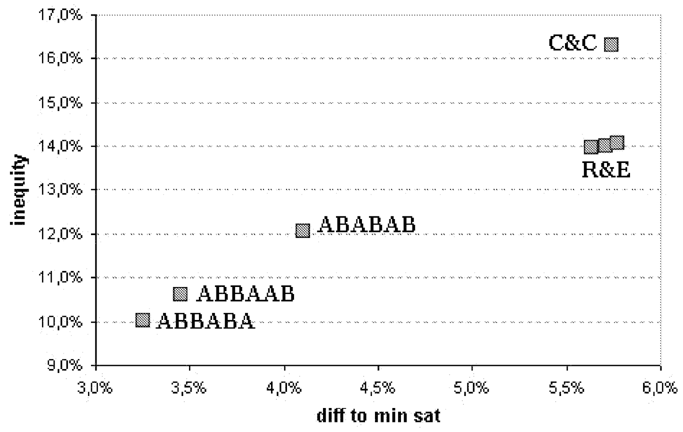

Figure 2: Min versus Equity

Thus at least in our examples minimum satisfaction correlates rather strongly with equity, but not with efficiency, so probably minimum satisfaction should still be considered a fairness measure.

3.1 Similar or opposite preferences

These averages shown in Figure 1 use the assumption that the preferences of the two players are independent. Interestingly, things change if we drop this assumption. A measure how close the preferences of the two players are is again the correlation coefficient, this time between the two lists of Ann's and Beth's values of the six pralines. It is close to -1 if both preferences are almost opposite, Ann prefering most what Beth prefers least, and close to 1 if both preferences are similar, if both players prefer the same pralines.

We ordered the data of these 2000 randomly generated preferences by correlation coefficient, grouped into 20 groups.

In Figure 3 the different efficiency values for three methods, C&C, ABABAB, and R&E12, are displayed, for the different correlation coefficients. Cut & Choose shows about the same efficiency in any case. The two other methods are more efficient for smaller correlation coefficient, i.e. for opposing preferences. For these opposing preferences, both these methods are slightly more efficient than Cut & Choose.

The pattern for equity is displayed in Figure 4. The more opposite the preferences are the higher the inequity is for Cut & Choose, but the lower the inequity for the other two methods.

As a conclusion, generally 'Cut & Choose' has the highest efficiency, and the three 'selecting step-by-step' methods highest equity while still achieving resonable efficiency. If the preferences of the two players are opposite, however, then 'Cut & Choose' seems to be a little inefficient and rather unfair, and the 'selecting step-by-step' methods seem to be best. The "Random & Exchange' methods also seem to produce rather efficient and fair outcomes for opposite preferences.

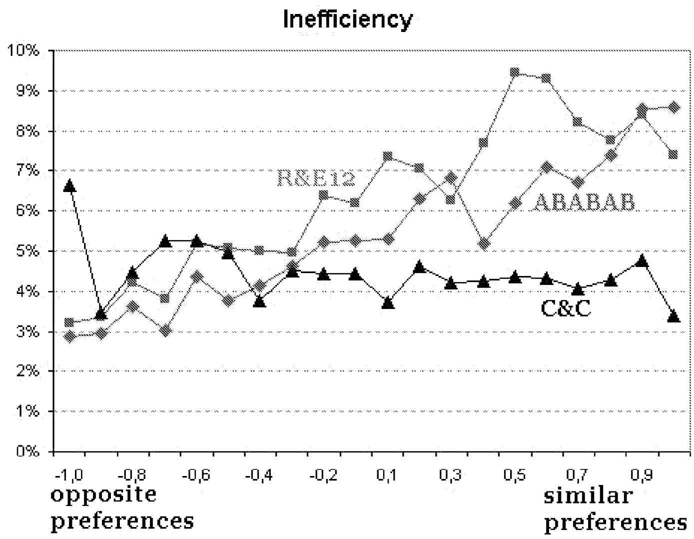

Figure 3: Efficiency of three methods for different correlation coefficients

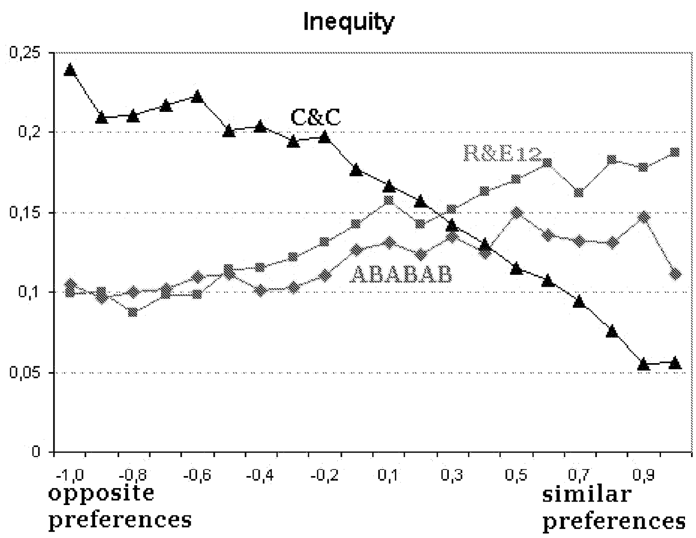

Figure 4: Equity of three methods for different correlation coefficients

4 Incomplete Information

What happens if Ann and Beth know the values they assign to the different pralines, but not their sister's values? This is called 'incomplete information'. Would the methods discussed above still work?

Still both players will have a belief on the other's preferences. We assume that both players believe that the other's preferences are independent from their own, and that the values are independent random variables with uniform distribution.

4.1 Choosing one by one

With these beliefs it is rather obvious how the players play the three games ABABAB, ABBABA, and ABBAAB: They would just play what we called 'greedy strategy' before: Each player chooses the item available having the largest value to her. There is no reason to play tactical and choose something of lesser value, since no prediction is possible about what the other player will choose next. More precisely, all remaining items could be chosen with equal probability by the other player.

4.2 Cut and Choose

The continuous version, dividing a cake, is usually introduced with this assumption of incomplete information. Then this continuous version is easy to play: Ann would cut the cake into two parts of equal value to her.

If Ann divides the six pralines into two heaps of three paralines each, it doesn't even matter what the cumulative value of each heap for Ann is. Since Beth's preferences are independent, she will choose any of the two heaps with probability 50%, therfore the expected value for Ann will be 50% in each case. She would still probably try to balance the values of two heaps as close as possible if she wants to maximize the worst case. But things are different for heaps of two versus four pralines. Then Beth is much more likely to select the heap of four. How much more likely is fairly complicated to compute, but she will choose the 4-pralines heap with probability 0.919444. That means that if Ann has two pralines that together are worth more than 50% of her total value, she may be pretty sure to receive this valuable heap and the end. Also, Beth will select a 5-pralines heap over the remaining one-praline heap with even much higher probability of about 0.998611. Therefore Ann's optimal strategy is to choose any division into 3 pralines versus 3 pralines—maybe one where the value of each heap is as close as possible for Ann— unless there the sum of the two highest valued pralines is more than 50% of the whole, or the highest values praline is more than 50% of the whole. In this case Ann would choose that 2 versus 4 division, or that 1 versus 5 division. If there are more than one of these divisions, she would choose the one having the highest expectation, where the expectation in case of two pralines of relative value of $a > 50\%$ would be $0.919444 \cdot a + 0.080556 \cdot (1 - a)$ and in case of one praline of relative value $b > 50\%$ it would be $0.998611 \cdot b + 0.001389 \cdot (1 - b)$. Any of these events happening would be rare.

4.3 Random and Exchange

Analyzing these games with incomplete information is difficult, and we will not try to do this in this paper. Obviously players would make proposals to exchange higher valued items for lower-valued ones, but initially they would not have a clue whether the other player would accept or not. And the player having to accept would probably accept if the exchange makes sense. That is, both players would initially play greedy and not tactical. However, after each round the players would learn something about the other's preferences. For the simulations of 'R&DE2' below we assume greedy play.

4.4 Simulations

We did run simulations of the games, except 'R&E12' and 'R&E2', with incomplete information on the same 2000 data sets of random independend values for the pralines with uniform distribution. Figure 5 repeats the values from Figure 1 for complete information in grey, and also shows the results for incomplete information as black diamonds. Surprisingly, the 'selecting step-by-step' methods don't differ much, but 'Cut & Choose' and "R&DE2' are dramatically less efficient for incomplete information. In 'Cut and Choose' the average payoff for Ann is 54.8%, down from 71.9% in the case where Ann and Beth know each other's preferences. Of course this is due to the fact that in most cases Ann just has to offer a distribution of 3 pralines versus 3 pralines, in which case Beth may take any of the two heaps with equal probability, and the expected payoff for Ann is just 50%. That it is higher than 50% is caused by the rare but occuring case of having two or one very valuable pralines, which was discussed above. Beth on the other hand profits from incomplete information. Her average payoff in the 2000 simulations were 64.9%, compared to 56.0% with complete information.

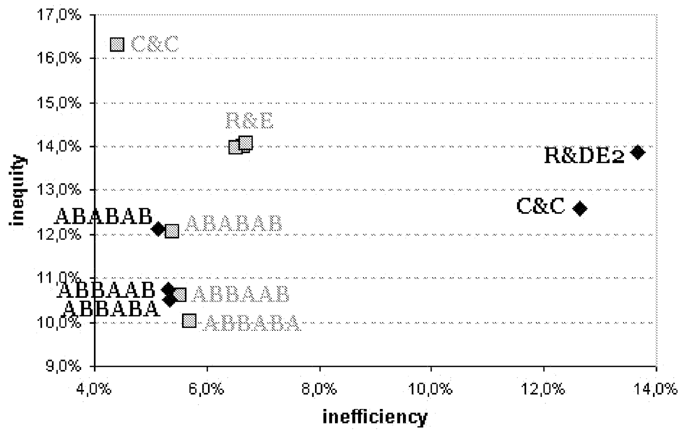

Figure 5: Efficiency versus Equity for incomplete information (in black)

Thus in the incomplete information case the three games 'ABABAB', 'ABBABA', and 'ABBAAB' are definitely better than the 'C&C' and 'R&E' games. They even yield more efficient outcomes than with complete information, but these differences are not statistically significant.

Although in real life transactions there is often the tendency to hide the own preferences in order to get a higher payoff in this way (taking the possibility of tactical play from the other player), **mutual** openness, transparency, is usually considered advantageous for a society. At least in our concrete situation, our simulations on the games considered do not exclude the possibility that in some cases it might be better if both players don't know the preferences of the other.

5 Pointers towards the literature

Distributing six goods among two persons is a very concrete case of a much more abstract question to divide goods among persons who value different parts of the goods differently. The classical paper for divisible goods is [S 1949] and [A 1982]. Indivisible goods were already mentioned in [S 1949] (see also the description in [S 1998]), and have been discussed in a number of papers since. In papers like [S 1949] or [ADG 1991], monetary side payments are allowed. In [BEF 2003], [HP 2002], and [BK 2005], the preferences that are ordinal, meaning that each player can order the pralines from best to worst, but they cannot tell how much better one tastes than another. In [BS 2006], [LMMS 2004] and part of [BD 2005] the Santa Claus optimization problem for known preferences is discussed. Papers like [HP 2002] or part of [BD 2005] discvuss the incomplete information case.

Acknowledgment: I would like to thank two anonymous referees for their remarks and suggestions.

References

[ADG 1991] A. Alkan, G. Demange, D. Gale, Fair allocation of indivisible goods and criteria of justice, *Econometrica* 59 (1991) 1023-1039.

[A 1982] A.K. Austin, Sharing a cake, *The Mathematical Gazette* 66 (1982) 212-215.

[BS 2006] N. Bansal, M. Sviridenko, The Santa Claus problem, Proceedings STOC 2006, 31-40.

[BD 2005] I. Bezáková, V. Dani, Allocating indivisible goods, *ACM SIGecom Exchanges* 5 (2005) 11-18.

[BEF 2003] Steven J. Brams, Paul H. Edelman, Peter C. Fishburn, Fair division of indivisible items, *Theory and Decision* 55 (2003) 147-180.

[BK 2005] S.J. Brams, D.R. King, Efficient fair division: Help the worst off or avoid envy? Rationality and Society, 17 (2005) 387-421. Department of Politics, New York University.

[DH 1988] S. Demko, T.P. Hill, Equitable distribution of individible items, *Mathematical Soc. Sciences* 16 (1988) 145-158.

[HP 2002] Dorothea Herreiner, Clemens Puppe, A simple procedure for finding equitable allocations of indivisible goods, *Soc. Choice Welfare* 19 (2002) 415-430.

[LMMS 2004] R.J. Lipton, E. Markakis, E. Mossel, A. Saberi, On approximately fair allocations of indivisible goods, *ACM Conference on Electronic Commerce* (2004) 125-131.

[S 1949] H. Steinhaus, Sur la division pragmatique, *Econometrica (supplement)* 17 (1949) 315-319.

[S 1998] H. Steinhaus, Mathematical Snapshots, Dover Publications (1999) p. 67-69.